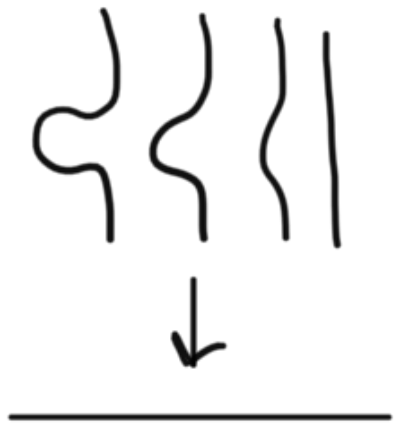
12th April 22

SMOOTH MORPHISMS

GOAL smooth mor. = "nicely varying family of very regular schemes"

"flat"

geometrically regular ($\Leftrightarrow$ smth fibers)



$$\begin{array}{c} X \\ \downarrow \\ Y \end{array}$$

reference is 2nd edition of Görtz / Wedhorn [GW]

DEF 6.14 $f : X \to Y$, $d \geq 0$ integer, $x \in X$

i, we call f is "smooth of rel. dim. d at x" if
$\exists$ affine open nbh U of x
$\exists$ " " $V = \operatorname{Spec} R$ of $f(x)$

$\exists$ open immersion φ

$$\begin{tikzcd} U \arrow[rr, hook, "\varphi"] \arrow[dr, "f|_U"'] & & \operatorname{Spec} \frac{R[T_1, \dots T_n]}{f_1, \dots, f_{n-d}} \arrow[dl, "\text{proj.}"] \\ & V = \operatorname{Spec} R & \end{tikzcd}$$

for suitable n & $f_1, \dots, f_{n-d}$.
s.th. the Jacobian

$$J(x) := \left(\frac{\partial f_i}{\partial T_j} (x) \right)_{i,j} \quad k(x)^{(n-d) \times n}$$

has rank $n-d$.

ii, f is "<u>smooth</u>" if smth at all pts.

iii, f "<u>étale</u>" if smooth of rel dim 0.

<u>EX</u>

① open immersions are étale

② $\mathbb{A}^n_{\mathbb{Z}} \longrightarrow \operatorname{Spec} \mathbb{Z}$ is smooth of rel dim n.

③ $\operatorname{Spec} \mathbb{C} \longrightarrow \operatorname{Spec} \mathbb{R}$

$$\begin{tikzcd}
\mathbb{C} & & \frac{\mathbb{R}[X]}{X^2+1} \arrow[ll, "\cong"'] \\
& \mathbb{R} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

minipo is seperable i.e. no multiple roots in $\mathbb{C}$ $\Rightarrow$ derivative nonzero.

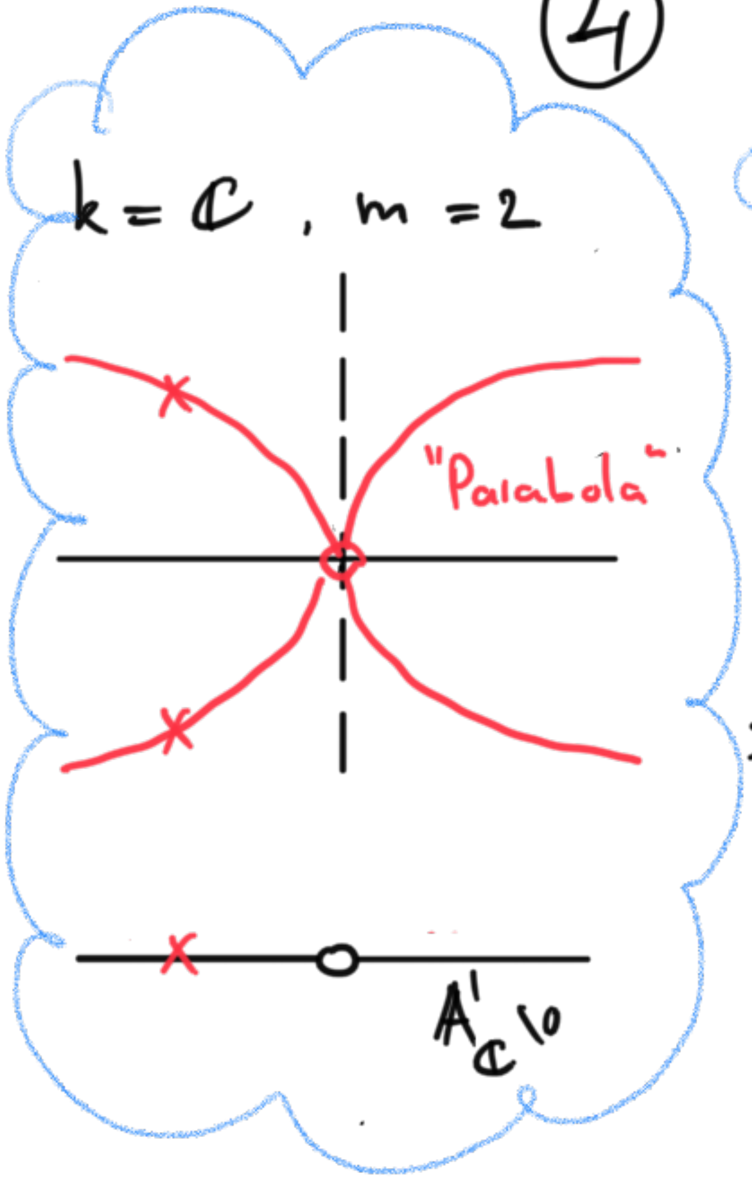
$J = 2X \Rightarrow J((i)) = 2i \neq 0$

$\Rightarrow$ étale

(*) "$\Leftarrow$" holds as well.

<u>in fact</u> $k \subseteq K$ field ext

fin. & seperable $\overset{(*)}{\Longrightarrow}$ $\operatorname{Spec} K \to \operatorname{Spec} k$



④ $k[T^{\pm 1}] \longrightarrow \dfrac{k[T^{\pm 1}, X]}{X^m - T} =: A$

k field of char $k \neq m$

$J = m X^{m-1}$

$x \in \operatorname{Spec} A$ s.th. $J(x) = 0$

$\Leftrightarrow X^{m-1} \in \mathfrak{p}_x \Leftrightarrow \underset{\underset{T}{\|}}{X^m} \in \mathfrak{p}_x \subseteq A$ ↯

$\Rightarrow$ étale mor.

BASIC PROP.

PROP 6.15 Smooth mor. are LOCS, LOCT BC & COMP

EX

- $\mathbb{A}^n_S \longrightarrow S$ is smooth

  S any Scheme

- $\mathbb{P}^n_S \longrightarrow S$ is smooth

PROP 6.15 (1) $f : X \longrightarrow Y$ & x is a smooth pt. of rel. dim d then $\exists$ open nbh U of x s.th.

$f|_U$ is smooth of rel dim d.

PROOF $\quad \operatorname{rank} J(x) = n-d =: r$

$$\begin{tikzcd}
x \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \\
\mathcal{U}' \arrow[r, hook] \arrow[dr] & \operatorname{Spec} \frac{R[T_1,\dots,T_n]}{f_1,\dots,f_{n-d}} \arrow[d] \\
& \operatorname{Spec} R
\end{tikzcd}$$

$\Longleftrightarrow \; \exists$ submatrix M of J with $\det M(x) \neq 0$

$\in R[T_1,\dots,T_n]$

pick $\mathcal{U} := D(\det M) \cap X$ does the job. q.e.d.

That is: $X_{sm} := \{x \in X \mid$ s.th. f is smth at $x\}$

"smooth locus" $\subseteq X$ open subscheme.

COMPLETIONS

DEF A ring, $\mathfrak{a} \subseteq A$ & M A-module

$$\hat{M}^{\mathfrak{a}} := \lim M/\mathfrak{a}^n M$$

"completion of M wrt $\mathfrak{a}$"

EX $\quad A = R[T_1, \dots, T_n] \qquad \mathfrak{a} = (T_1, \dots, T_n)$

$$\hat{A}^{\mathfrak{a}} = R[[T_1, \dots, T_n]]$$

$$\lim \frac{R[T]}{T^n} = \left\{ (f_n) \in \prod \frac{R[T]}{T^n} \text{ s.th. } f_n = f_{n+1} \bmod T^n \right\}$$

$$= R[[T]]$$

$S := A \setminus \mathfrak{p}$

PROP a) $\hat{A}^{\mathfrak{a}}$ — prime $= \left(\widehat{A_{\mathfrak{a}}}\right)^{\mathfrak{a} A_{\mathfrak{a}}}$ $\qquad A_{\mathfrak{a}} = S^{-1}A$

(as ref. B.39) b) A Noeth & $M' \to M \to M''$ ex. seq. of fin. gen. modules then

$$\hat{M'}^{\mathfrak{a}} \longrightarrow \hat{M}^{\mathfrak{a}} \longrightarrow \hat{M''}^{\mathfrak{a}}$$

is exact again.

c) $\hat{A}^{a} \otimes_A M \longrightarrow \hat{M}^{a}$ is an iso. (M fin gen)

EX

- $m = (X_1, \dots, X_n) \in \mathbb{A}^n_k$

$$\hat{\mathcal{O}}^{m}_{\mathbb{A}^n_k, m} \overset{a)}{=} \left(k[X_1, \dots, X_n]\right)^{\wedge m} = k[[T_1, \dots, T_n]]$$

- $m = (X_1, \dots, X_n) \in X = V(I) \subseteq \mathbb{A}^n_k$

$$\hat{\mathcal{O}}^{m}_{X,x} \overset{a), b), c)}{=} \dots = \frac{k[[X_1, \dots, X_n]]}{I}$$

PROP 6.23 k field & X scheme$/_k$ k

$x \in X_k(k)$ smooth of rel dim d.

then $\hat{\mathcal{O}}^{m_x}_{X,x} \cong k[[X_1, \dots, X_d]]$

PROOF local: wlog $X = \dfrac{k[T_1, \dots, T_n]}{f_1, \dots, f_{n-d}}$

x k-rat: wlog x origin

i.e. $(T_1, \dots, T_n)$

wlog $J = \begin{pmatrix} \boxed{J'} & \cdots \end{pmatrix}$ $r \times r$

$r := n-d$

& rank $J' = r$

$$k[[U_1, \dots, U_n]] \xrightarrow{\phi} k[[T_1, \dots, T]]$$

$$U_i \longmapsto \begin{cases} f_i, & i = 1, \dots, r \\ T_i, & \text{else} \end{cases}$$

has Jacobian

$$J_\phi = \begin{pmatrix} \boxed{J'} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$J_\phi(0)$ has full rank & is thus invertible

Lem 6.22 $\Longrightarrow$ ϕ isomorphism

hence

$$\begin{array}{ccc} \dfrac{k[[U_1, \dots, U_n]]}{U_1, \dots, U_{n-d}} & \xrightarrow{\phi} & \dfrac{k[[T_1, \dots, T_n]]}{f_1, \dots, f_{n-d}} \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ k[[\underbrace{U_{r+1}, \dots, U_n}_{d \text{ variables}}]] & & \hat{\mathcal{O}}^{m}_{X,x} \end{array}$$

q.e.d.

EX (first non example!)

$$Z := \operatorname{Spec} \frac{k[X,Y]}{XY} \ni m := (X,Y)$$

$$\hat{\mathcal{O}}_{Z,m}^{m} = \frac{k[[X,Y]]}{XY}$$ has zero divisors.

$\Rightarrow \varphi$ not smooth over k.

SMOOTHNESS VS. REGULARITY

DEF X loc. Noeth. scheme $x \in X$ is "regular" $(\mathcal{O}_{X,x}, m)$ is regular i.e.

$$\dim \mathcal{O}_{X,x} = \dim_{k(x)} m/m^2$$

LEM 6.26 k a field & X loc. of finite over k $x \in X$ smooth pt of rel. dim d over k then

$\mathcal{O}_{X,x}$ is regular &

$$\dim \mathcal{O}_{X,x} = \begin{cases} d, & \text{if } x \text{ closed} \\ \leq d, & \text{else.} \end{cases}$$

PROOF Local problem, so wlog

$$X \hookrightarrow k[T_1, \dots, T_n] \big/ f_1, \dots, f_{n-d} \subseteq \mathbb{A}^n_k$$

$$x \longmapsto y$$

$$\mathcal{O}_{X,x} = \mathcal{O}_{\mathbb{A}^n, y} \big/ f_1, \dots, f_{n-d}$$

(A) case x closed & $k(y) = k$.

$\operatorname{rank} J(x) = n-d \Rightarrow f_1, \dots, f_{n-d}$ are linearly indep. in m_y / m_y^2

$\overset{B.77}{\Longleftrightarrow} \mathcal{O}_{\mathbb{A}^n, y} / f_1, \dots, f_{n-d}$ is regular

of $\dim = \dim \mathcal{O}_{\mathbb{A}^n, y} - (n-d) = d$

(B) x closed & $k(y)$ arbitrary

$$\begin{tikzcd}
X \times_{\operatorname{Spec} k} \operatorname{Spec}(k(y)) \arrow[r, hook] \arrow[d] & \mathbb{A}^n_{k(y)} \arrow[r] \arrow[d, "\phi", "y' \mapsto y"'] & \operatorname{Spec} k(y) \arrow[d] \\
X \arrow[r, hook, "x \mapsto y"] & \mathbb{A}^n_k \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

pick y' over y closed

& denote $m_{y'} \underset{\max}{\subseteq} k(y)[T_1, \dots, T_n]$

$$\begin{tikzcd} k[T_1, \dots, T_n] \arrow[d, "\phi"] \\ k(y)[T_1, \dots, T_n] \end{tikzcd}$$

Ⓐ $\Rightarrow \phi(f_1), \dots, \phi(f_{n-d})$ are l.i. in $m_{y'}/m_{y'}^2$

But we have

$$\begin{tikzcd} m_y \arrow[r, "\phi"] \arrow[d] & m_{y'} \arrow[d] \\ m_y/m_y^2 \arrow[r] & m_{y'}/m_{y'}^2 \end{tikzcd}$$

k(y) - linear

$\Rightarrow f_1, \dots, f_{n-d}$ are l.i. in m_y/m_y^2

$\underset{B.77}{\Longrightarrow}$ $\mathcal{O}_{X,x}$ is regular & of dim d.

(c) x is arbitrary

X loc. of fin. type$/_k$ $\overset{(*)}{\Longrightarrow}$ closed pts are dense

(*) See Prop. 3.35 in [GW]

$\Longrightarrow$ pick $x \rightsquigarrow x'$ closed

Then $\mathcal{O}_{X,x}$ is a loc. of $\mathcal{O}_{X,x'}$

thus $\mathcal{O}_{X,x}$ is regular &

of dim $\le d$. q.e.d.

EX (another non example)

$$\operatorname{Spec} \mathbb{F}_p(T) \xrightarrow{\varphi} \operatorname{Spec} \mathbb{F}_p(T^p)$$

p prime number

This is the case of a <u>not</u> seperable field extension.

Base change:

$$\begin{tikzcd} P \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{F}_p(T) \arrow[l] \\ \mathbb{F}_p(T) \arrow[u] & \mathbb{F}_p(T^p) \arrow[l] \arrow[u] \end{tikzcd}$$

<u>Claim</u> $\quad P \cong \dfrac{\mathbb{F}_p(T)[X]}{X^p}$

Indeed $\quad \mathbb{F}_p(T) = \dfrac{\mathbb{F}_p(T^p)[Y]}{Y^p - T^p}$

Thus $\quad \mathbb{F}_p(T) \otimes_{\mathbb{F}_p(T^p)} \dfrac{\mathbb{F}_p(T^p)[Y]}{Y^p - T^p}$

$$= \frac{\mathbb{F}_p(T)[Y]}{Y^p - T^p} = \frac{\mathbb{F}_p(T)[Y]}{(Y-T)^p}$$

$$= \frac{\mathbb{F}_p(T)[X]}{X^p}$$

$\overset{\text{Lem 6.26}}{\Longrightarrow}$ ϕ is <u>not</u> smooth.

Spec $\dfrac{\mathbb{F}_p(T)[X]}{X^p}$ is <u>not</u> regular e.g. via Jacobi criterion

~~**LEM 6.27**~~ THM k field & $X = V(g_1, \dots, g_s) \subseteq \mathbb{A}^n_k$
x is a closed point s.th.

$$\operatorname{rank} J(x) = n - \dim \mathcal{O}_{X,x}$$

~~then~~ iff x is smooth.

We get a good picture of the relation of smoothness and regularity:

THM 6.28

X k-scheme loc. of finite type, $x \in X$ closed, $k \subseteq K$ an algebraically closed field ext. then t.f.a.e.

(*) $X_K = X \times_{\operatorname{Spec} k} \operatorname{Spec} K$

i) $X \longrightarrow \operatorname{Spec} k$ is smooth of rel. dim d at x

$\Downarrow$ BC

ii) $\forall$ pts $\bar{x} \in X_K$ over x: $X_K \to \operatorname{Spec} K$ is smth of rel. dim. d. at $\bar{x}$

(*)

$\Downarrow$ almost Prop 6.23

iii) — " — : $\hat{\mathcal{O}}_{X_K,\bar{x}} \cong K[[X_1, \dots, X_d]]$

iv) — " — : $\mathcal{O}_{X_K,\bar{x}}$ is regular & has dim d

If these are satisfied, then

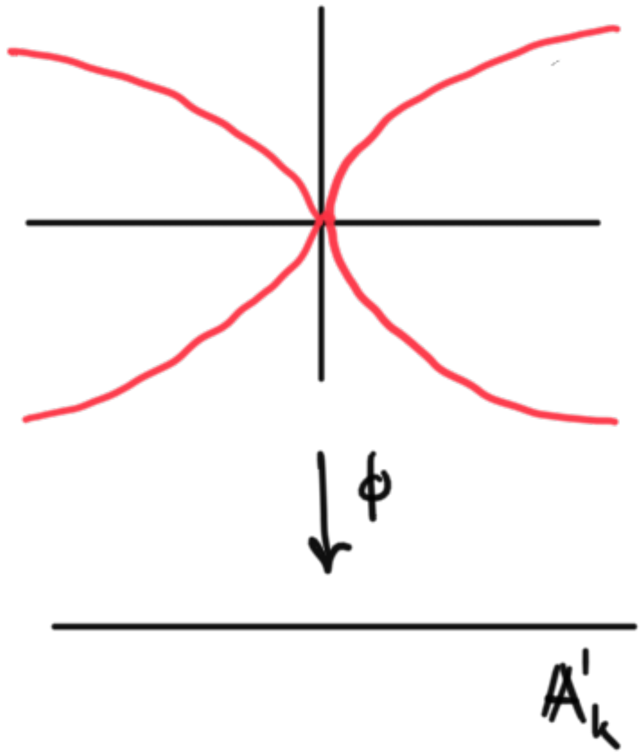
v) x is regular & $\dim \mathcal{O}_{X,x} = d$

Lemma 6.27 + Jacob. criterion for regularity

Lem. 6.26

Furthermore if $k(x) = k$ then v_1 implies all others.

<u>EX</u> Revisiting



$$k[T] \xrightarrow{\phi} k[T,X]/X^m - T =: A$$

Is ϕ smooth at $p = (X,T)$?

No, consider base change

$$\begin{tikzcd} k[T] \arrow[r] \arrow[d] & A \arrow[d] \\ k[T]/T \arrow[r] & k[X]/X^m \end{tikzcd}$$

and $k[X]/X^m$ not regular.

We thus know for $X := \operatorname{Spec} A \to \mathbb{A}^1_k$

$$X_{sm} = D(T) \subsetneq X$$

Not in the talk

<u>COR 6.32</u> For a k-scheme X loc of finite type then t.f.a.e.

i) X smooth over k

ii) $\forall$ field ext $k \subseteq L$: X_L regular

iii) $\exists$ alg closed ext $k \subseteq K$: X_K regular.

this is called "geometrically regular"

FLATNESS

THM 14.24 Smooth morphisms are flat.

Putting everything we have done together

smooth mor ...

by Cor 6.32 this is eq. to geometr. regular fibers.

⇔ ... · have smooth fibers

... · are flat

... · are locally finitely presented
(didn't discuss that, but not to hard to see.)

The converse is true as well

SP 01V8: $f: X \longrightarrow Y$ flat w. smooth fibers & locally of finite presentation then f is smooth.